The Prediction of Axisymmetric Turbulent Swirling Boundary Layers

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Axisymmetrical swirling boundary layers and fully three-dimensional layers have in common a two-component shear stress. Since the former are amenable to treatment by two-dimensional computational methods, they constitute useful and economical testing terrain for advanced turbulence models. Traditional "effective-viscosity" based turbulence models, arbitrarily extended to account for the influence of a swirl component of velocity, display a serious lack of universality. A high Reynolds number turbulence model is developed in the present study which provides algebraic equations for all six Reynolds stresses. The model and its mixing-length derivative are both employed to predict a variety of swirling boundary layers. A promising improvement in universality of predictive power is exhibited. The principal conclusion is that future research should concentrate on extending the applicability of stress turbulence models into the sublayer region since the effective-viscosity based models are least satisfactory there.

Nomenclature

C_1, C_2 = constants in the pressure-strain model
C_D = constant relating the dissipation rate ε to the length scale l
C_B, C_S, C_W, q = constants in the length-scale equation
d = pipe diameter of freejet flow
D = dissipation
Df = diffusion
ε = dissipation rate of turbulence energy
k = $\frac{1}{2}(\overline{v_1^2} + \overline{v_2^2} + \overline{v_3^2})$, kinetic energy of turbulence
l = length scale of turbulence
l_m = Prandtl mixing length
p = pressure
P = production
Ps = pressure-strain
Q = $2\pi \int_0^\delta V_1\, dy/\Omega r^2$, radial volumetric flow rate for disk
r = radial distance from axis of symmetry
R = radius of cylinder
Re = $\Omega r^2/\nu$, Reynolds number for disk
S = $\int_0^\delta r^2 V_1 V_3\, dy \Big/ \frac{d}{2} \int_0^\delta r(V_1^2 - V_3^2/2)\, dr$, swirl number for freejet
v = fluctuating velocity
$\mathbf{v}$ = instantaneous velocity vector
V = time-average velocity
$\overline{v_i v_j}$ = time-average double velocity correlation, Reynolds stress
x_i = coordinates, $i = 1, 2, 3$
x = coordinate in the main flow direction
y = cross-stream distance measured normal to a surface
α = geometrical angle in Fig. 1
β = function in the Monin-Oboukhov formula
δ = boundary-layer thickness, distance from surface where velocity is 0.01 of its maximum value
δ_{ij} = Kronecker delta
$\delta_{2,x}$ = $\int_0^\delta \frac{V_1}{V_{1,\infty}}\left(1 - \frac{V_1}{V_{1,\infty}}\right) dy$, momentum thickness in axial direction
$\delta_{2,\theta}$ = $\int_0^\delta (V_1 V_3/\Omega R V_{1,\infty})\, dy$, momentum thickness in the circumferential direction
κ, λ = mixing-length constants
η = nondimensional radial distance for freejet
μ = viscosity
ν = μ_{lam}/ρ
ρ = density
σ = constant in the modelling of diffusion terms
$\sigma_{2,3}$ = viscosity ratio
τ = shear stress
Ω = rotational speed

Subscripts

1 = the predominant flow direction
2 = the cross-stream direction
3 = the circumferential direction
1,2 = plane in the main and cross-stream directions
2,3 = plane in the circumferential and cross-stream directions
k = turbulence energy
kl = product of turbulence energy and length scale
lam = laminar
m = maximum value
s = slot or pipe exit
∞ = freestream

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1. Introduction

1.1 Scientific and Engineering Importance

SWIRLING boundary layers are those possessing a circumferential or rotational component of mean velocity in a surface normal to the direction of the streamwise flow. The present paper is concerned with the subclass of swirling turbulent boundary layers having symmetry with respect to a single axis of rotation. Flows belonging to this class are described by reference to the axial and radial coordinates only, all of their properties being invariant with respect to the circumferential coordinate. In consequence, they may be computed by existing, reliable and rapid, *two-dimensional* computational methods.

The scientific interest in such boundary layers arises from the fact that they possess two significant shear stress components, namely $-\rho\overline{v_1 v_2}$ and $-\rho\overline{v_2 v_3}$. In this respect they are akin to three-dimensional layers, and as such they represent a useful intermediate stage of study in the development of turbulence models for fully-three-dimensional layers.

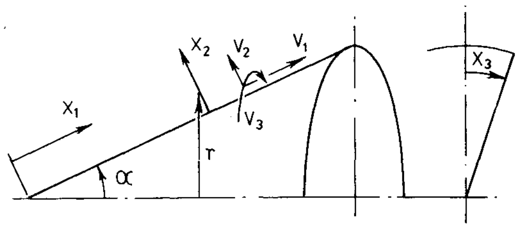

Fig. 1 The co-ordinate system.

The cooling of rotating electrical machinery, the cooling of gas-turbine disks, and the performance of swirl-stabilized furnace burners are three important related engineering problems. It is true that, in these examples, the flows are normally recirculating in nature, while the present work is concerned only with boundary-layer flows. But although the computational problems of solving the governing partial differential equations for recirculating swirling flows have been overcome,[1] the turbulence modelling of recirculating flows, even in the absence of swirl, is a little explored subject.

1.2 Brief Summary of Previous Theoretical Work

All of the early prediction methods for swirling boundary layers were of the integral-profile kind and as such they were restricted to simple geometries, constant-property fluids, and simple boundary conditions, see for example Refs. 2 and 3. In 1967 Koh and Price[4] successfully employed a method of the much more powerful finite-difference variety to predict the swirling boundary layer near a rotating cone. This work was followed by that of Bayley and Owen[5] and Owen[6] who applied the original finite-difference method of Patankar and Spalding[7] to predict the turbulent boundary layers which form on the stator and rotor of a parallel-disk system with outflow. The present authors have utilized a more recent version of the same method, also described in Ref. 7, to obtain predictions near free disks, cylinders and cones, and between parallel disks[8]; Lilley[9] similarly obtained predictions of free axisymmetrical swirling jets.

Although Cham and Head[10–12] have recently predicted with some success the flow near a rotating disk, cylinder, and nose-body using an integral-profile procedure, there can be little doubt that finite-difference methods, are the only ones of sufficient potential to be of real interest to design engineers. Given the availability of reliable numerical methods, the task of providing adequate models for turbulence simulation remains. Launder and Spalding[13] review the current status of turbulence modelling in a recent book which they have devoted to this subject. The abovementioned studies[5–9] have employed Prandtl's mixing-length hypothesis which falls into the least sophisticated class of turbulence models.

1.3 Purpose and Scope of the Present Study

This paper reports the results of a doctoral project, the aim of which was the development of turbulence models for the prediction of axisymmetrical swirling boundary layers.

Our early studies were based on Prandtl's mixing-length concept, suitably adapted to swirling flow by several plausible, yet arbitrary extensions to account for the influence of the circumferential flow. Unfortunately, the empirical constants of the mixing-length formulations exhibited an inadequate degree of universality across the range of experimental conditions considered. A more sophisticated turbulence model was then tested in which the turbulence was characterized by the turbulence kinetic energy and a length scale, the values of these quantities being obtained from the solution of their own differential equations.[14] The predictions displayed no over-all improvement over those based on the mixing-length concept. The central reason for the failure of these two models, it was concluded, is that swirling flows exhibit a significant anisotropy of the turbulent viscosity,[9,15,16] the effects of which cannot be satisfactorily allowed for by any self-evident arbitrary extension of turbulence models which serve to predict nonswirling boundary layers.

Attention was turned to the modelling of the transport equations for the Reynolds stresses or double-velocity correlations themselves and the results of these latest studies are reported herein.

2. Analysis

2.1 The Mean Flow Equations

The boundary-layer forms of the momentum equations in the coordinate system illustrated in Fig. 1 are

x_1-direction momentum

$$\rho V_1 \frac{\partial V_1}{dx_1} + \rho V_2 \frac{\partial V_1}{dx_2} = -\frac{1}{r}\frac{\partial}{\partial x_2}(r\rho\overline{v_1 v_2}) - \frac{\partial p}{\partial x_1} + \rho\frac{V_3^{\,2}}{r}\sin\alpha \quad (1)$$

x_2-direction momentum

$$0 = -\partial p/\partial x_2 + \rho(V_3^{\,2}/r)\cos\alpha \quad (2)$$

x_3-direction momentum

$$\rho V_1 \frac{\partial(rV_3)}{\partial x_1} + \rho V_2 \frac{\partial(rV_3)}{\partial x_2} = -\frac{1}{r}\frac{\partial}{\partial x_2}(r^2\rho\overline{v_2 v_3}) \quad (3)$$

2.2 The Turbulence Equations

a) The Approach

It is required to find relations, for the unknown Reynolds tangential stresses $-\rho\overline{v_1 v_2}$ and $-\rho\overline{v_2 v_3}$ which, when combined with the mean-flow equations (1) and (3), result in a closed and soluble set. The present approach to the problem is based on an analysis of the transport equations for the Reynolds stresses themselves.

Hanjalic and Launder[17] and du P. Donaldson[18] have recently proposed and tested turbulence models of this variety for two-dimensional, *nonswirling* boundary layers. The aforementioned authors solve differential equations for the turbulence kinetic-energy k, turbulence dissipation rate ε, and the Reynolds stress $-\rho\overline{v_1 v_2}$; the latter author and his colleagues solve differential equations for k and $-\rho\overline{v_1 v_2}$, but they employ an algebraic formulation for ε and their modelling of the pressure-strain correlation is not complete.

b) Equations for the Double Correlations

For the problem of swirling boundary layers, our starting point for the derivation of the transport equations for the Reynolds stresses or double-velocity correlations is the Navier-Stokes equations, which for steady flow, may be expressed as

$$\underbrace{(\mathbf{v}\cdot\nabla)\mathbf{v}}_{1} = -\underbrace{(1/\rho)\nabla p}_{2} + \underbrace{\nu\nabla^2\mathbf{v}}_{3} \quad (4)$$

For the geometry of Fig. 1 we need these transport equations in a general orthogonal coordinate system. The amount of manipulation needed to do this is by no means trivial and we follow short cuts that take account of the physical nature and Navier-Stokes parentage of the terms which finally appear in the equations for the double correlations.

Term 1 of Eq. (4) gives rise to terms which express convection, production and diffusion of the double-velocity correlations; term 2 leads to pressure diffusion and pressure strain terms; while term 3 leads to diffusion and viscous dissipation terms. The present approach involves operating only on term 1. The ensuing terms for the convection and production of the Reynolds stresses are extracted and the diffusion terms which remain are associated with the diffusion contributions of terms 2 and 3. The diffusion, pressure-strain and dissipation contributions of terms 2 and 3 are modelled directly, their constituent terms are not formally determined. Prior to the modelling, but after invoking the boundary-layer approximations, with x_1, x_2, and x_3 corresponding, respectively, to the main stream, cross-stream and circumferential directions, there result the following equations for the double correlations in axisymmetrical, curvilinear, orthogonal coordinates

$$V_1 \frac{\partial \overline{v_1^2}}{\partial x_1} + V_2 \frac{\partial \overline{v_1^2}}{\partial x_2} = 4\frac{V_3}{r}\overline{v_1 v_3}\frac{\partial r}{\partial x_1} - 2\overline{v_1 v_2}\frac{\partial V_1}{\partial x_2} + (Df + Ps + D)_{1,1} \quad (5)$$

$$V_1 \frac{\partial \overline{v_2^2}}{\partial x_1} + V_2 \frac{\partial \overline{v_2^2}}{\partial x_2} = 4\frac{V_3}{r}\overline{v_2 v_3}\frac{\partial r}{\partial x_2} + (Df + Ps + D)_{2,2} \quad (6)$$

$$V_1 \frac{\partial \overline{v_3^2}}{\partial x_1} + V_2 \frac{\partial \overline{v_3^2}}{\partial x_2} = -4\frac{V_3}{r}\overline{v_1 v_3}\frac{\partial r}{\partial x_1} - 4\frac{V_3}{r}\overline{v_2 v_3}\frac{\partial r}{\partial x_2} - 2\overline{v_2 v_3}\, r\frac{\partial (V_3/r)}{\partial x_2} + (Df + Ps + D)_{3,3} \quad (7)$$

$$V_1 \frac{\partial \overline{v_1 v_2}}{\partial x_1} + V_2 \frac{\partial \overline{v_1 v_2}}{\partial x_2} = -\overline{v_2^2}\frac{\partial V_1}{\partial x_2} + 2\frac{V_3}{r}\overline{v_2 v_3}\frac{\partial r}{\partial x_1} + 2\frac{V_3}{r}\overline{v_1 v_3}\frac{\partial r}{\partial x_2} + (Df + Ps + D)_{1,2} \quad (8)$$

$$V_1 \frac{\partial \overline{v_1 v_3}}{\partial x_1} + V_2 \frac{\partial \overline{v_1 v_3}}{\partial x_2} = -\overline{v_1 v_2}\, r\frac{\partial (V_3/r)}{\partial x_2} - \overline{v_2 v_3}\frac{\partial V_1}{\partial x_2} - 2v_1 v_2 \frac{V_3}{r}\frac{\partial r}{\partial x_2} - 2\frac{V_3}{r}(\overline{v_1^2} - \overline{v_3^2})\frac{\partial r}{\partial x_1} + (Df + Ps + D)_{1,3} \quad (9)$$

$$\underbrace{V_1 \frac{\partial \overline{v_2 v_3}}{\partial x_1} + V_2 \frac{\partial \overline{v_2 v_3}}{\partial x_2}}_{\text{convection}} = \underbrace{-\overline{v_2^2}\, r\frac{\partial (V_3/r)}{\partial x_2} - 2\frac{V_3}{r}\overline{v_1 v_2}\frac{\partial r}{\partial x_1} - 2\frac{V_3}{r}(\overline{v_2^2} - \overline{v_3^2})\frac{\partial r}{\partial x_2} + (Df + Ps + D)_{2,3}}_{\text{production} + Df + Ps + D} \quad (10)$$

where Df = diffusion, Ps = pressure-strain, and D = dissipation.

c) The Modelled Stress Equations

It is the small scale eddies which are primarily responsible for the turbulence dissipation. For high turbulence Reynolds numbers, they are isotropic even though there may be anistropy of the large scale motion. Consequently, we may express the dissipation as[17]

$$D_{ij} = -\tfrac{2}{3}\delta_{ij}\varepsilon \quad (11)$$

where ε is the turbulence energy dissipation rate.

The pressure-strain or "redistribution" correlations arise from two physical sources: the mutual interaction of the fluctuating velocities, and the interaction of the mean rate of strain with the turbulence.[17] Most authors have adopted Rotta's[19] proposal to model the first mentioned part of the correlation, and we have conformed to this practice. Rotta has also proposed a representation of the second mentioned part, as have Hanjalic and Launder,[17] and Naot, Shavit, and Wolfshtein.[20] None of these have been much tested however, so we chose to adopt the formulation of the last authors because of its simplicity. Hence, the full form of our pressure-strain representation becomes

$$Ps_{ij} = \underbrace{C_1\, \varepsilon/k(\overline{v_i v_j} - \tfrac{2}{3}k\,\delta_{ij})}_{\text{1: Rotta}^{19}} - \underbrace{C_2(P_{ij} - \tfrac{2}{3}P_k\,\delta_{ij})}_{\text{2. Naot and co-workers}^{20}} \quad (12)$$

where P_{ij} represents the production terms of Eqs. (5–10), and $P_k = \tfrac{1}{2}(P_{1,1} + P_{2,2} + P_{3,3})$ stands for the production of turbulence energy.

The differential nature of Eqs. (5–10) can be very conveniently eliminated by extending the modelling one stage further through the adoption of a further, and not unreasonable, suggestion of Rodi[21] that the local convection less the diffusion of a Reynolds stress is proportional to the convection less the diffusion of turbulence kinetic energy k in the ratio of the value of the stress to the value of the kinetic energy. This approximation permits Eqs. (5–10) to be represented by

$$(P_k - \varepsilon)\overline{v_i v_j}/k = (Ps + P + D)_{ij} \quad (13)$$

where it has been recognized that the convection minus the diffusion of k is equal to the production minus the dissipation of k [see Sec. 2.2(d) below]. When the modelled expressions (11) and (12) are inserted, there results an algebraic equation for the Reynolds stresses

$$\overline{v_i v_j} = \left[\frac{k}{P_k + \varepsilon(C_1 - 1)}\right]\left[\tfrac{2}{3}\delta_{ij}\{C_2 P_k + \varepsilon(C_1 - 1)\} + P_{ij}(1 - C_2)\right] \quad (14)$$

Algebraic formulae for all six Reynolds stresses may be generated from Eq. (14). The stress production terms P_{ij} are the corresponding ones of Eqs. (5–10) and they are expressed wholly in terms of the stresses themselves and known mean-flow quantities. Consequently, closure of the set of stress equations necessitates only the additional knowledge of the production P_k of the turbulence energy and of the turbulence dissipation rate ε. We obtain this information by solving the transport equation for k (defined as one half the sum of the normal stresses) and Rotta's[19] "kl" length-scale equation; dimensional arguments lead to the relation

$$l = C_D k^{3/2}/\varepsilon \quad (15)$$

where l has the dimension of a length-scale and C_D is a further constant to be determined.

d) The Turbulence Energy and Length-Scale Equations

Summation of Eqs. (5–7) for the normal stresses, and division by two gives the equation for k as

$$\underbrace{V_1 \frac{\partial k}{\partial x_1} + V_2 \frac{\partial k}{\partial x_2}}_{\text{convection}} = \underbrace{-\left(\overline{v_1 v_2}\frac{\partial V_1}{\partial x_2} + \overline{v_2 v_3}\, r\frac{\partial (V_3/r)}{\partial x_2}\right)}_{\text{production}} + \underbrace{Df_k}_{\text{diffusion}} + \underbrace{D_k}_{\text{dissipation}} \quad (16)$$

It should be noted that the pressure-strain terms vanish from the equation for turbulence kinetic energy. Following the practice suggested in Ref. 13 for nonswirling two-dimensional boundary layers, we presume that the diffusion of k obeys a gradient-type relation

$$Df_k = \frac{1}{r}\frac{\partial}{\partial x_2}\left(r\frac{k^{1/2} l}{\sigma_k}\frac{\partial k}{\partial x_2}\right) \quad (17)$$

where σ_k is a parameter which is expected to be close to unity. The dissipation of k, which is half the sum of the dissipation for the normal stresses, is from Eq. (11) simply equal to $-\varepsilon$. With the diffusion and dissipation so modeled, the final form of the equation for k is

$$\underbrace{V_1 \frac{\partial k}{\partial x_1} + V_2 \frac{\partial k}{\partial x_2}}_{\text{convection}} = \underbrace{-\overline{v_1 v_2}\left(\frac{\partial V_1}{\partial x_2}\right) - \overline{v_2 v_3}\left(r\frac{\partial (V_3/r)}{\partial x_2}\right)}_{\text{production}} + \underbrace{\frac{1}{r}\frac{\partial}{\partial x_2}\left(r\frac{k^{1/2} l}{\sigma_k}\frac{\partial k}{\partial x_2}\right)}_{\text{diffusion}} - \underbrace{\varepsilon}_{\text{dissipation}} \quad (18)$$

It is not unlikely that equations for separate length scales, pertaining to each of the coordinate directions will ultimately require to be solved in order to handle swirling flows with sufficient generality. We are supposing here, however, that the anisotropic swirl flow eddies may be characterized by just one "effective" length scale. The form of Rotta's equation which we solve is, see Ref. 22

$$\underbrace{V_1 \frac{\partial kl}{\partial x_1} + V_2 \frac{\partial kl}{\partial x_2}}_{\text{convection}} = \underbrace{\frac{1}{r}\frac{\partial}{\partial x_2}\left(r\frac{k^{1/2} l}{\sigma_{kl}}\frac{\partial kl}{\partial x_2}\right)}_{\text{diffusion}} - \underbrace{C_B l\left(\overline{v_1 v_2}\frac{\partial V_1}{\partial x_2} + \overline{v_2 v_3}\, r\frac{\partial (V_3/r)}{\partial x_2}\right)}_{\text{production}} - \underbrace{C_S k^{3/2}}_{\text{dissipation}} - \underbrace{C_W(l^{-1}\kappa y C_D^{1/4})^{-q} k^{3/2}}_{\text{wall-damping}} \quad (19)$$

Following the practice for the k equation, the diffusion transport is again represented by a gradient-type law; σ_{kl} is a constant of order unity. The "wall-damping" term was proposed by Ng and Spalding[23] and found by these authors to result in better predictions for nonswirling flows. This term is, of course, absent for the case of freeflows.

Table 1 Values of turbulence model constants of present study compared with those of other workers

	C_1	C_2	C_B	C_S	C_D	σ_k	σ_{kl}	C_W	q	κ
Wall flows without swirl[a]										
Ng and Spalding[23]			0.84	0.055	0.10	2	1.2	0.056	4	0.4
Ng[24]			0.98	0.058	0.09	1	1	0.078	4	0.4
Hanjalic and Launder[17]	2.8									
Flow near a rotating cylinder										
Present study	2.8	0.4,[b] 0.2[c]	0.98	0.058	0.09	1	1	0.078	4	0.4
Free shear flows without swirl										
Rotta[25]	2.8									
Naot and coworkers[20]		0.8								
Hanjalic and Launder[17]	2.8									
Round freejet without swirl										
Rodi and Spalding[26]			0.98	0.0397	0.055	1	0.3			
Rodi[21]	2.5	0.4	0.98	0.0397	0.055	1	1			
Free swirling jet										
Lilley[9]			0.98	0.0397	0.055	1	1			
Present study	2.8	0.4, 0.5[c]	0.98	0.0397	0.055	1	1			

[a] These studies have all been based on plane geometries; possible radius effects in our rotating cylinder study, will however be negligible since $\delta/R \ll 1$.
[b,c] Values of C_2, referred to in the text as $C_{2,n}$ and $C_{2,t}$, applicable to normal and tangential double correlations, respectively.

e) *The Empirical Constants*

The complete Reynolds-stress model comprising the six algebraic stress relations represented by Eq. (14) along with Eq. (18) and (19), contain 10 empirical constants: C_1, C_2, C_B, C_S, C_D, C_W, q, σ_k, and σ_{kl}. We have used the model to predict the free swirling jet and rotating cylinder test flows, and we have been guided in our choice of the values of the constants by the values found by other authors to give good predictions for related, but nonswirling flows.

Unfortunately, even without swirl, changes in the constants between the wall flows and the round freejet flows are required as Table 1 reveals. At best, one might hope that the introduction of swirl would not give rise to any further lack of universality; whatever the case, the nonswirling flow constants offer the only available indication of the starting point for swirl-flow computations.

The constant C_2, which appears in the turbulence model for the second part of the pressure-strain terms was, on the basis of tentative evidence, assigned the value of 0.8 by the proposers of the model,[20] whereas Rodi[21] has found the value of 0.4 to be more appropriate for a round nonswirling jet. Because of the uncertainty in the modelling, we have permitted this parameter to assume either of two values, $C_{2,n}$ or $C_{2,t}$, depending on whether it occurs in an equation for a normal or a tangential double correlation. Both values have been determined by computer optimization and they turn out to be rather close to Rodi's value of 0.4. The assigning of two values for C_2 is an unappealing feature of the present turbulence model, but it represents a useful stop-gap measure in the absence of better knowledge for the simulation of the pressure-strains terms.

f) *A Mixing-Length Formulation*

It is of interest to deduce a simpler model of turbulence from the stress model previously presented. If a local balance between the production and dissipation of k is supposed (sometimes termed local-equilibrium) for nonswirling flow, Prandtl's mixing-length formula emerges.[13] For swirling flow, when the same assumption is applied to Eq. (18), the result is

$$\mu_{1,2}\left(\frac{\partial V_1}{\partial x_2}\right)^2 + \mu_{2,3}\left(r\frac{\partial(V_3/r)}{\partial x_2}\right)^2 = \rho\varepsilon \tag{20}$$

where the effective viscosities are defined in terms of the two main Reynolds stresses as

$$\mu_{1,2} \equiv -\rho\overline{v_1 v_2}\Big/\frac{\partial V_1}{\partial x_2} \tag{21}$$

and

$$\mu_{2,3} \equiv -\rho\overline{v_2 v_3}\Big/ r\frac{\partial(V_3/r)}{\partial x_2} \tag{22}$$

Now dimensional considerations reveal that a characteristic value of viscosity is proportional to $\rho l^{4/3}\varepsilon^{1/3}$; we may then arbitrarily write

$$\mu_{1,2} = \rho l_m^{4/3}\varepsilon^{1/3}$$

where l_m is a mixing length referenced to the $\mu_{1,2}$ viscosity. If for convenience we define a "viscosity ratio"

$$\sigma_{2,3} \equiv \mu_{1,2}/\mu_{2,3} \tag{23}$$

Eq. (20) may be rewritten as

$$\mu_{1,2} = \rho l_m^2\left[\left(\frac{\partial V_1}{\partial x_2}\right)^2 + \frac{1}{\sigma_{2,3}}\left(r\frac{\partial(V_3/r)}{\partial x_2}\right)^2\right]^{1/2} \tag{24}$$

This simple mixing-length formulation can be used for the prediction of swirling flows if the functional relationships for l_m and $\sigma_{2,3}$ are known. We presume away from walls, the following simple mixing-length distribution

$$\lambda\delta/\kappa < y \leqq \delta, \quad l_m = \lambda\delta \tag{25}$$

where κ and λ are adjustable constants to be determined by computer optimization. An analysis of the distribution of $\sigma_{2,3}$ remote from walls is provided in subsection (*g*) which follows; near walls the variation of $\sigma_{2,3}$ emerges as part of a modified mixing-length treatment described in subsection (*h*) below.

Predictions are presented in Sec. 3 using the mixing-length formulation for the free rotating disk flow. The computer optimized values of κ and λ are given there.

g) *The Viscosity Ratio*

Manipulation of Eq. (14) for the main stresses $-\rho\overline{v_1 v_2}$ and $-\rho\overline{v_2 v_3}$, with the definitions (21–23), leads to the following relation for the viscosity ratio:

$$\sigma_{2,3} = \frac{1 - \dfrac{2}{\overline{v_2^2}}\dfrac{V_3/r}{\partial V_1/\partial x_2}\left[\overline{v_2 v_3}\dfrac{\partial r}{\partial x_1} + \overline{v_1 v_3}\dfrac{\partial r}{\partial x_2}\right]}{1 + \dfrac{2}{\overline{v_2^2}\, r}\dfrac{V_3/r}{\partial(V_3/r)/\partial x_2}\left[\overline{v_1 v_2}\dfrac{\partial r}{\partial x_1} + (\overline{v_2^2} - \overline{v_3^2})\dfrac{\partial r}{\partial x_2}\right]} \tag{26}$$

It is interesting to consider the behavior of $\sigma_{2,3}$ for two special cases: flows for which the mainstream direction is parallel to the axis of symmetry such as a swirling jet or a rotating cylinder in an axial stream; and flows for which the mainstream direction is normal to the axis of symmetry, e.g., a rotating disk.

For the former, $x_2 = r$ and $\partial r/\partial x_1 = 0$. Further, the correlation $-\rho\overline{v_1 v_3}$ is the stress $\tau_{1,3}$. Since this stress is small for laminar boundary layers of the kind considered here, it seems reasonable to expect that it is also small when such layers are turbulent; Eq. (26) then reduces to

$$\sigma_{2,3} = (1-\beta Ri)^{-1} \quad (27)$$

β is defined by $[(\overline{v_3^2}/\overline{v_2^2})-1]$, and Ri is a Richardson number defined as

$$Ri \equiv 2V_3/r/r[\partial(V_3/r)/\partial x_2] \quad (28)$$

Equation (28) can be recognized as a relative of the empirical Monin-Oboukhov formula which modifies the length scale[27,28] when a body force acting normal to the streamline direction exists. The equation reveals that, when the body force is due to swirl, the modification is more properly applied to the viscosity ratio $\sigma_{2,3}$.

For the latter of the aforementioned special cases, $x_1 = r$ and $\partial r/\partial x_2 = 0$. Also fully turbulent flow will occur for a value of r which is large compared with the boundary-layer thickness, so V_3/r will be much less than $\partial V_1/\partial x_2$ and $r\,\partial(V_3/r)/\partial x_2$. Now the correlations $\overline{v_1 v_2}$, $\overline{v_2 v_3}$, and $\overline{v_2^2}$ may be expected to be of the same order, consequently we surmise that $\sigma_{2,3}$ is probably near to unity.

h) Near-Wall Region

It has been stated that the stress model described herein is restricted to regions where the local turbulence Reynolds number is high. Near walls, where this condition is not satisfied, it is the practice for nonswirling, two-dimensional boundary layers to employ wall laws, most often derived from the established "log-law" velocity relation. Now, in some swirling pipe-flow experiments, Backshall and Landis[29] have demonstrated that the resultant velocity obeys the conventional log-law and that the axial and swirl velocity components can be obtained by straightforward resolution. In some early calculations, we therefore determined the Couette-layer shear-stress components from the resolved log-law; there resulted boundary-layer predictions which were widely at variance with the data.[14]

Subsequently, some trial mixing-length based predictions were obtained by integrating the boundary-layer equations across the Couette-layer to the wall with the aid of van Driest's damping function.[30] It was inferred that the shear-stress and velocity vectors "twist" significantly relative to each other within the sublayer. Since Backshall and Landis were unable to measure very near the wall, the occurrence of this phenomenon in their flow was not revealed. However, Johnston[31] has more recently performed some very careful measurements in a three-dimensional boundary layer which clearly demonstrate the misalignment of these two vectors within the sublayer.

On the basis of this evidence, intuition led to our final procedure near walls in which the two viscosities $\mu_{1,2}$ and $\mu_{2,3}$, and consequently the Reynolds shear stresses through the constitutive relations (21) and (22), were determined from the following two modified mixing-length-based expressions

$$\mu_{1,2} = \rho\kappa^2 y^2\{1-\exp[-y(\tau_{1,2})^{1/2}/26\mu_{\rm lam}]\}^2\left[\left(\frac{\partial V_1}{\partial x_2}\right)^2 + \frac{1}{\sigma_{2,3}}\left(r\frac{\partial(V_3/r)}{\partial x_2}\right)^2\right]^{1/2} \quad (29)$$

$$\mu_{2,3} = \rho\kappa^2 y^2\{1-\exp[-y(\tau_{2,3})^{1/2}/26\mu_{\rm lam}]\}^2\left[\left(\frac{\partial V_1}{\partial x_2}\right)^2 + \frac{1}{\sigma_{2,3}}\left(r\frac{\partial(V_3/r)}{\partial x_2}\right)^2\right]^{1/2} \quad (30)$$

The modification pertains to the use of the local values of the directional shear stresses, $\tau_{1,2}$ and $\tau_{2,3}$ in the respective damping functions. These near-wall expressions were found to offer the key to the successful prediction of the range of test flows. It should be noted that their combination with the momentum equations, and constitutive relations forms a closed set from which $\sigma_{2,3}$ can be evaluated.

3. Comparison of Predictions with Experimental Results

3.1 The Free Swirling Jet

Strictly, the Reynolds-stress turbulence model proposed in Sec. [2b)–2e)] is applicable to two-dimensional high Reynolds number boundary layers. The only flow which does not violate any of these restrictions is a free, fully turbulent, axisymmetrical swirling jet for which the degree of swirl is not sufficient to cause recirculation; unfortunately here, as in many other areas of fluid mechanics today, the potential of computer-based prediction methods has outstripped the supply of established experimental data. We have made comparisons with the experimental results of Pratte and Keffer[32] and of Chigier and Chervinsky.[33] The former authors have measured all six double-correlations in a swirling jet using hot-wire techniques, a task which is by no means easy; and these authors have remarked[34] that one should not attempt to rely on their data for more than trends and magnitudes.

In Fig. 2 the predicted Reynolds stresses are compared with those measured by Pratte and Keffer at stations 6 and 12 diameters downstream from injection, respectively. Comparisons for the stress $\overline{v_1 v_3}$ are not shown because there is now some doubt surrounding the measured values.[34] With the notable exception of the stress $\overline{v_2 v_3}$ at 6 diameters downstream, the agreement is on the whole quite good.

Also shown in Fig. 2 are predicted profiles of the viscosity ratio $\sigma_{2,3}$. It is noted that a rather large maximum value of about 3 occurs at the axis for $x/d = 6$, and that this maximum falls off rapidly with downstream distance due to the rapid decay of the swirl component of velocity. The radially-averaged values of $\sigma_{2,3}$ are roughly the same as those found by Lilley[9] by computer optimization to give satisfactory predictions of mean quantities. It should be remarked that the degree of anisotropy of turbulence displayed by the predicted normal stresses is less than that which the values of the viscosity ratio $\sigma_{2,3}$ would suggest. The latter quantity appears, therefore, to exaggerate the actual level of anisotropy, emphasising the fact that the "effective viscosity" concept is a rather bad one for turbulent swirling flows.

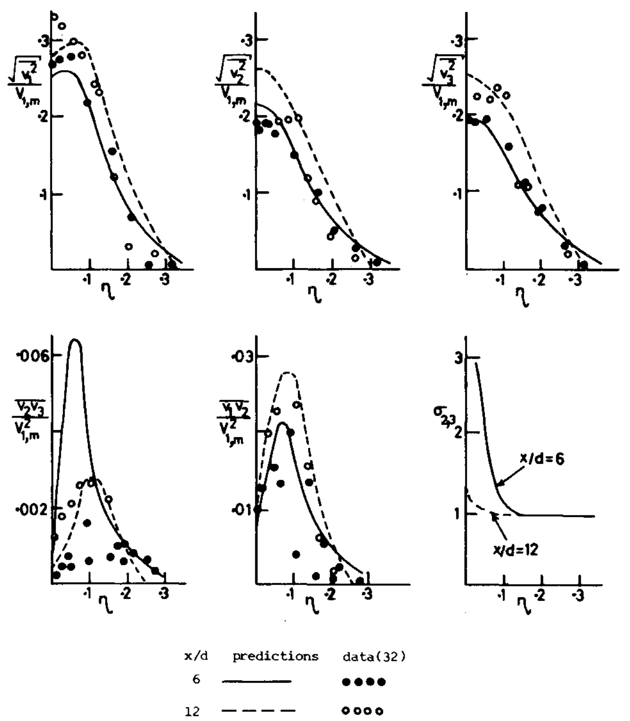

Fig. 2 Free swirling jet. Predictions of Reynolds stresses for a swirl number of 0.3.

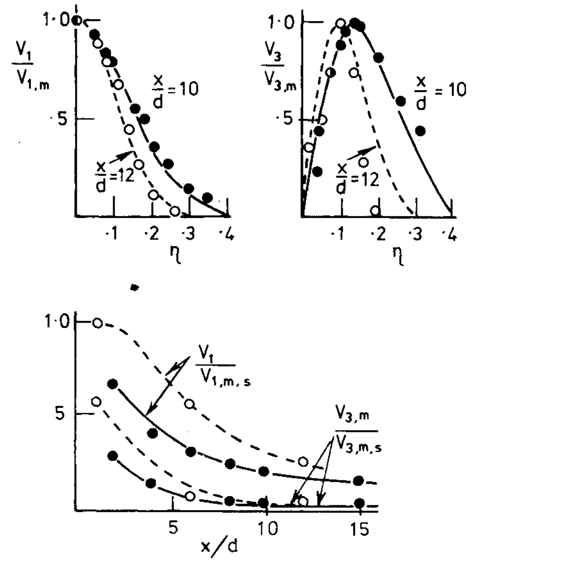

Fig. 3 Free swirling jet. Predictions of velocity profiles and velocity decay obtained using the stress model, for two values of the swirl number *S*.

Figure 3 shows some predictions of mean velocity distribution and mean-velocity decay compared with the measurements of Pratte and Keffer, and also with those of Chigier and Chervinsky. Since the Reynolds stresses are reasonably well predicted, not unexpectedly the predictions of mean quantities are also quite good. The decay of both the axial and swirl velocities are well predicted and this is particularly heartening since the less sophisticated effective viscosity based turbulence models do not possess this universality.[9] The prediction of the circumferential velocity V_3 for the Pratte and Keffer experiment stands out as being in poor agreement with the data. The V_3 profile of Chigier and Chervinsky is however well predicted.

3.2 Swirling Flows Near Walls

a) Rotating Disk

Erian and Tong[35] have measured the stresses $\overline{v_1 v_3}$, $\overline{v_1^2}$, and $\overline{v_3^2}$ near a free rotating disk. Using this data in the stress relations (14) alone, the remaining stresses: $\overline{v_1 v_2}$, $\overline{v_2 v_3}$, and $\overline{v_2^2}$ can be determined, along with the ratio $(1-C_{2,n})$ upon $(1-C_{2,t})$, without the need to specify the value of any of the adjustable constants. The outcome of this exercise is presented in Fig. 4 for a rotational Reynolds number of 9.93×10^5. The nearness of the ratio $(1-C_{2,n})/(1-C_{2,t})$ to unity is an indication of the validity of the turbulence model. The viscosity ratio $\sigma_{2,3}$ has also been calculated and it is, as expected from the arguments of Sec. 2.2(g), near unity; but in contrast to the freejet situation, the normal stresses now indicate significant anisotropy of turbulence.

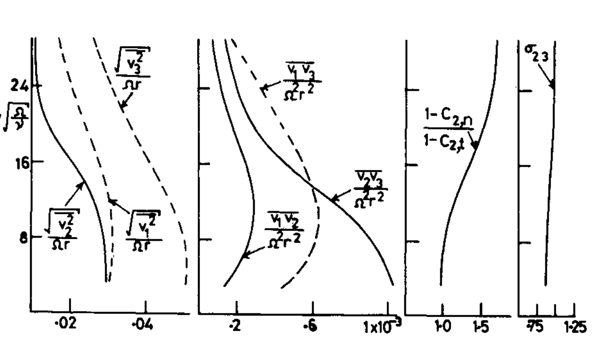

Fig. 4 Free rotating disk. Predictions obtained using the stress relations.

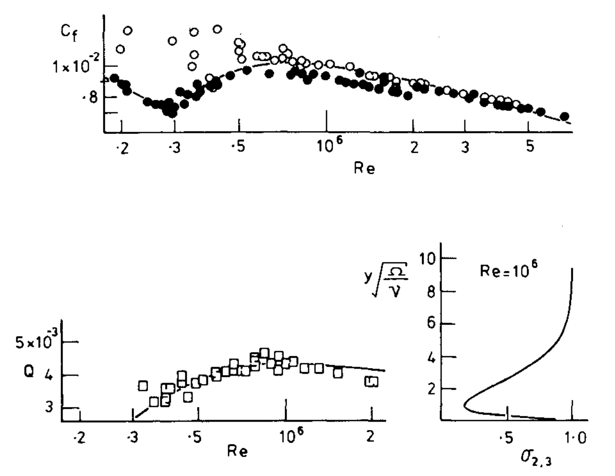

Fig. 5 Free rotating disk. Predictions of mean-flow quantities using the mixing-length model with the near-wall relations.

Since $\sigma_{2,3}$ is near unity in the outer and fully turbulent part of the boundary layer, we have felt justified in using the mixing-length formulation of Sec. 2.2(f) with $\sigma_{2,3}=1$, combined with the near-wall relations of Sec. 2.2(h) to procure mean flow predictions across the entire width of the boundary layer. Comparisons were made with the data of Eqs. (6, 10, and 36) and a selection of these is shown in Fig. 5. The optimum values found for κ and λ were 0.34 and 0.085. The former value is surprisingly low and it may be that a relatively smaller adjustment to the van Driest damping constant would result in a more usual value of κ being appropriate. We have preferred, however, in these mixing-length predictions to adjust only κ and λ with the damping constant fixed at its usual "flat-plate value" of 26.

The computed variation of $\sigma_{2,3}$ in the near-wall region is also shown and it is seen to depart substantially from unity. Yet good predictions of the disk flow are known to be possible using mixing-length formulations which presume an isotropic viscosity all the way to the wall, for somewhat different values of κ and λ.[8,37] The reason must be that since $V_{3,\max} \gg V_{1,\max}$, the flow is dominated by the $\mu_{2,3}$ viscosity alone.

b) Rotating Cylinder

Mean flow data for a cylinder rotating in a coaxial stream have been provided by Parr[38] and by Furuya and his co-workers.[39] Predictions of the growths of the axial and circumferential momentum thicknesses, obtained using the Reynolds stress turbulence model matched near the wall to the relations (29) and (30), are shown compared with the data in Fig. 6. The values of the constants of the stress model are those recorded in Table 1 for the full range of data. In the near-wall expressions (29) and (30) two values of the constant κ, 0.43 and 0.48, were, respectively, required to predict the data of Refs. 38 and 39. The enhanced universality of the present near-wall treatment is revealed by the predictions, also displayed, obtained assuming $\sigma_{2,3}=1$ (i.e., isotropic viscosity) in this region.

Although neither Parr nor Furuya made measurements of any turbulence quantities, the predictions of these quantities are nonetheless of interest. Figure 7 shows the predicted profiles for one of Parr's conditions of the turbulence kinetic energy, length scale, Reynolds stresses, ratio of dissipation to production of energy, and the viscosity ratio. Two aspects of these predictions are worthy of note. Firstly, the ratio of dissipation to production $(\varepsilon/P)_k$ is nowhere far from unity, as would be expected for a near-equilibrium flow, so that the rather approximate way in

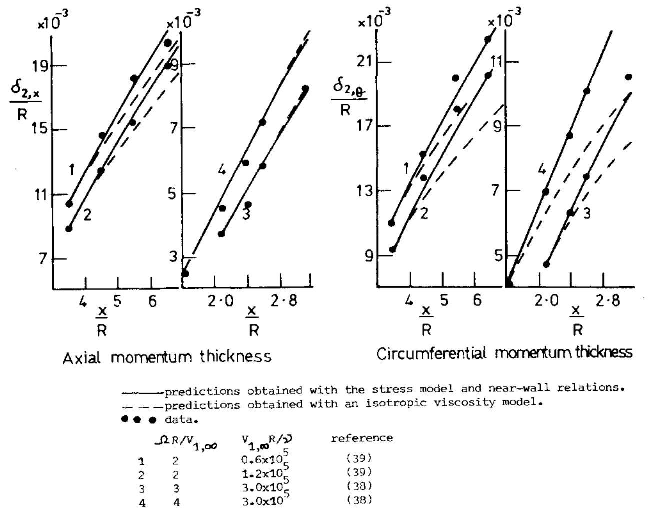

Fig. 6 Cylinder rotating in a uniform axial stream. Predictions of mean-flow quantities.

which the convection and diffusion contributions to the stress equations have been modeled should not be of great consequence. Secondly, the stress ratio $\sigma_{2,3}$ is near unity where the local Reynolds number of turbulence is high, as surmised in Sec. 2.2(g), but the departure from unity within the sublayer is again considerable. We can conclude that, since both ε/P and $\sigma_{2,3}$ are close to unity in the outer part of the boundary layer, application of the mixing-length formulation in place of the stress model in this region would not result in a loss of universality; computations have been performed which substantiate this.

4. Conclusions

1) Two-dimensional swirling boundary layers are not amenable to universal prediction by turbulence models which presume a scalar "effective" viscosity.

2) In consequence, turbulence models which provide equations from which Reynolds stresses themselves can be calculated are required.

3) A model of this kind, applicable to boundary-layer flows where the local Reynolds number of turbulence is high, is proposed in the present study.

4) The stress model is used to predict two experimental swirling free jet flows, for one of which measurements of turbulence quantities are reported. The agreement between the predictions and the data is fairly good.

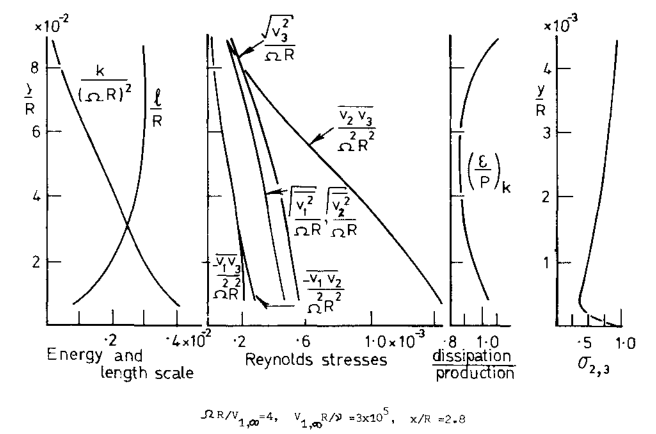

Fig. 7 Cylinder rotating in a uniform axial stream. Predictions of turbulence quantities.

5) A mixing-length formulation which arises naturally from the stress model is presented. However, the ratio of the effective viscosities in the mainstream and swirl directions, $\sigma_{2,3}$, appears in this formulation.

6) For a free swirling jet $\sigma_{2,3}$ is found to depart significantly from unity only near the axis, and less than about 10 diam downstream.

7) Examination of the equations of the stress model suggests that $\sigma_{2,3}$ is near unity in the fully turbulent outer part of swirling boundary layers near walls; computations confirm this.

8) In the sublayer very near a wall, the resultant velocity vector "twists" relative to the resultant shear-stress vector; resolved shear stresses cannot therefore be obtained from the log-law resultant velocity profile.

9) A fairly universal determination of the mainstream and circumferentially-directed viscosities in the sublayer appears possible using mixing-length based expressions modified by van Driest damping terms which differ in each of the two directions by the use of the respective local values of the shear stress.

10) Predictions of the boundary layers near a rotating cylinder and disk are obtained using the Reynolds stress and mixing-length model, respectively, matched to the modified mixing length based expressions near the wall. The agreement with the experimental data is, in all cases, more universal than that given by former arbitrary effective-viscosity models. The predictions reveal that $\sigma_{2,3}$ departs significantly from unity in the sublayer.

11) Much further development and testing of the Reynolds stress turbulence model is required. In particular, its extension to the near wall region where $\sigma_{2,3}$ is not close to unity is of especial importance if the goal of universal application and truly unified treatment is to be attained.

12) There is a serious shortage of reliable data for turbulence quantities in swirling flows; these are required for the further development of turbulence models.

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